

A dynamical condition for differentiability of Mather's average action

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Abstract

We prove the differentiability of β of Mather function on all homology classes corresponding to rotation vectors of measures whose supports are contained in a Lipschitz Lagrangian absorbing graph, invariant by Tonelli Hamiltonians. We also show the relationship between local differentiability of β and local integrability of the Hamiltonian flow.

1 Introduction

Given a Tonelli Lagrangian L , Mather introduced the β -function of L , which is a convex and superlinear function. Many interesting properties of the Euler–Lagrange flow can be derived from the study of the behaviour of the β -function. Understanding whether or not this function is differentiable and what are the implications of its regularity to the dynamics of the system is an interesting problem. This type of problem was developed by D. Massart in several works as, for example, [14] and [15].

Even in this context, D. Massart and A. Sorrentino get in the work *Differentiability of Mather's average action and integrability on closed surfaces* (see [16]) the relation, on closed surfaces, between the differentiability of β -function and the integrability of the system. However there are examples of systems which are not integrable but have invariant Lipschitz Lagrangian graphs, i.e. invariant graphs of the form $\mathcal{G}_{\eta,u} = \text{graph}(\eta + du)$ where η is a closed one-form and u is a function of class C^1 with Lipschitz differential.

Motivated by these problems, in this work we study the differentiability of β at homologies h whose the measures with vector rotation h are supported on an invariant

Lipschitz Lagrangian graph. We obtain differentiability of β in these homologies if the invariant graph is an absorbing graph, i.e. a graph which not contain ω -limit of minimizing curves out of it¹. More precisely, we prove the following theorem:

Theorem 1 *Let $\mathcal{G}_{\eta,u}$ be an invariant Lipschitz Lagrangian graph. Then $\mathcal{G}_{\eta,u}$ is absorbing if and only if β is differentiable at h for all $h \in \partial\alpha([\eta])$ and $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$.*

One can derive some consequences of this result. For instance, if the system is locally Lipschitz integrable on an invariant Lipschitz Lagrangian graph $\mathcal{G}_{\eta,u} \subset T^*M$, i.e. there exists a neighborhood \mathcal{V} of $\mathcal{G}_{\eta,u}$ in T^*M foliated by disjoint invariant Lipschitz Lagrangian graphs, of course that the graphs contained in \mathcal{V} are absorbing, so the following result is a local version of a result of D. Massart and A. Sorrentino (See [16], Lemma 5).

Theorem 2 *Let $\mathcal{G}_{\eta,u}$ be an invariant Lipschitz Lagrangian graph. If H is locally Lipschitz integrable on $\mathcal{G}_{\eta,u}$, then there exists a neighborhood $U_0 \subset H^1(M; \mathbb{R})$ of $[\eta]$ such that β is differentiable at any point of $V = \bigcup_{c \in U_0} \partial\alpha(c)$.*

We prove the converse of Theorem 2 in the case M equals torus \mathbb{T}^2 (see Theorem 15). In this case, the set $V = \bigcup_{c \in U_0} \partial\alpha(c)$, obtained in the above statement, is open in $H_1(\mathbb{T}^2; \mathbb{R})$. Then we generalize ([16], Theorem 3) to local case.

We also give a particular attention to existence of neighborhood contained in the tiered Mañé, introduced by M-C. Arnaud in [1], and its relation with the local integrability of system and therefore with the local differentiability of β . Indeed, we prove a local version of a result of M-C. Arnaud (See [2], Theorem 1), in the Section 5, Corollary 13.

2 Preliminaries

Let M be a compact connected manifold and TM its tangent bundle. A Tonelli's Lagrangian is a function $L : TM \rightarrow \mathbb{R}$ of class at least C^2 which is convex and superlinear. Let us recall the main concepts introduced by Mather in [17]. Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ -algebra on TM which are invariant under the Euler–Lagrange flow φ_t^L . The Euler Lagrange flow generated by L does not change by

¹the formal definitions and all notations are defined in the Sections 2, 3 and 4

adding a closed one form η and the action of a probability measure $\mu \in \mathcal{M}(L)$, defined by

$$A_{L-c}(\mu) = \int_{TM} (L - \eta) d\mu$$

depends only on the cohomology class $c = [\eta] \in H^1(M; \mathbb{R})$.

The minimal action value, which also depends only on the cohomology class $c = [\eta]$, is denoted by $-\alpha(c)$, that is:

$$\alpha(c) = - \inf_{\mu \in \mathcal{M}(L)} A_{L-c}(\mu).$$

Mather proved that the function $c \mapsto \alpha(c)$, so-called α of Mather function, is convex and superlinear. It is known that $\alpha(c)$ is the energy level that contains the *Mather set for the cohomology class c*:

$$\widetilde{\mathcal{M}}_c = \overline{\bigcup_{\mu} \text{supp}(\mu)},$$

where the union is taken over the set of Borel probability measures $\mu \in \mathcal{M}(L)$ called *c-minimizing*, i.e. $\alpha(c) = -A_{L-c}(\mu)$. The set $\widetilde{\mathcal{M}}_c$ is a compact invariant set which is a graph over a compact subset \mathcal{M}_c of M , the projected Mather set (see [17]). \mathcal{M}_c is laminated by curves, which are global (or time independent) minimizers.

Given a probability measure $\mu \in \mathcal{M}(L)$, its *homology* or its *rotation vector* is defined as the unique $\rho(\mu) \in H_1(M; \mathbb{R})$ such that

$$\langle \rho(\mu), [\omega] \rangle = \int_{TM} \omega d\mu,$$

for all closed 1-forms ω on M . By convexity, we can consider the dual Fenchel of α , called β of Mather function, as

$$\beta(h) = \inf_{\rho(\mu)=h} A_L(\mu).$$

Mather also proved that the β function is convex and superlinear. We say that a measure $\mu \in \mathcal{M}(L)$ with $\rho(\mu) = h$ is *h-minimizing* if $\beta(h) = A_L(\mu)$. The set $\widetilde{\mathcal{M}}^h$ is the union of supports of probability measures *h-minimizing*.

In general, the maps α and β are neither strictly convex, nor differentiable. The projection on domain of regions of graph where either map is affine are called flats. Actually, if the map is strictly convex at a point, the flat is this only point and if the map is not strictly convex, the flat is non-trivial. By duality we have the inequality

$$\alpha(c) + \beta(h) \geq \langle c, h \rangle, \forall c \in H^1(M; \mathbb{R}), \forall h \in H_1(M; \mathbb{R}),$$

called Fenchel inequality. Given $c \in H^1(M; \mathbb{R})$ (resp. $h \in H_1(M; \mathbb{R})$), the homology class $h \in H_1(M; \mathbb{R})$ (resp. $c \in H^1(M; \mathbb{R})$) achieving equality in the Fenchel inequality is called subderivative of α in c (resp. subderivative of β in h). The set composed by subderivatives of α in c (resp. subderivatives of β in h) is called Legendre transform of c (resp. h), and denoted $\partial\alpha(c)$ (resp. $\partial\beta(h)$). Therefore, $\partial\alpha(c)$ is a flat of β and $\partial\beta(h)$ is a flat of α . By convexity, the sets $\partial\alpha(c)$ and $\partial\beta(h)$ are non-empty.

Many interesting properties of the Euler–Lagrange flow can be derived from the study of the behaviour of the β -function. For instance, if h is an *extremal point* of β -function, i.e. h is not convex combination of two elements in a same flat of β , then there exist ergodic measures with homology h (see [13]).

Let us recall that we can associate to such a Tonelli’s Lagrangian L the Hamiltonian function $H : T^*M \rightarrow \mathbb{R}$ via Legendre transform $\mathcal{L} : TM \rightarrow T^*M$, which under our assumption, is a diffeomorphism of class at least C^1 , defined in coordinates by

$$\mathcal{L}(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v) \right).$$

Actually, H is the dual Fenchel of L and also is convex and superlinear. Given a cohomology class c and a closed 1-form η_c with $[\eta_c] = c$, we consider the Hamilton-Jacobi equation

$$H(x, \eta_c(x) + d_x u) = \alpha(c). \quad (\text{HJ})$$

A Lipschitz function $u : M \rightarrow \mathbb{R}$ is called a subsolution of Hamilton-Jacobi for the Lagrangian $L - c$ if for some closed 1-form η_c with $[\eta_c] = c$, we have

$$H(x, \eta_c(x) + d_x u) \leq \alpha(c), \quad (1)$$

at almost every point. Note that this definition is equivalent to the notion of viscosity subsolutions (see [9]). We denote by $C^{1,1}$ the set of differentiable functions with Lipschitz differential. Observe that a $C^{1,1}$ function u is solution of (HJ) if and only if the graph of $\eta_c + du$, denoted by $\mathcal{G}_{\eta_c, u}$, is invariant under Hamiltonian flow.

We now recall the definition of calibrated curves (see [9]). If $u : M \rightarrow \mathbb{R}$ is a subsolution of Hamilton-Jacobi for $L - c$, we say that the curve $\gamma : I \rightarrow M$ is $(u, L - c, \alpha(c))$ -calibrated if, for the representative η_c of the cohomology class c given in (1), we have the equality

$$u(\gamma(t)) - u(\gamma(t')) = \int_{t'}^t L(\gamma(s), \dot{\gamma}(s)) - \eta_c(\dot{\gamma}(s)) + \alpha(c) ds,$$

for all $t', t \in I$. The subset $\widetilde{\mathcal{I}}_c(u)$ of TM is defined by

$$\widetilde{\mathcal{I}}_c(u) = \{(x, v) \in TM : \gamma_{(x, v)} \text{ is } (u, L - c, \alpha(c))\text{-calibrated}\},$$

where $\gamma_{(x,v)} = \pi \circ \varphi_t^L(x, v)$. The set $\tilde{\mathcal{I}}_c(u)$ is invariant and the curves contained in it are called *curves c-minimizing*.

Using the sets $\tilde{\mathcal{I}}_c(u)$, one can give (see [10]) the following characterization of the Mañé set and of the Aubry set:

$$\tilde{\mathcal{N}}_c = \bigcup_{u \in SS_c} \tilde{\mathcal{I}}_c(u) \text{ and } \tilde{\mathcal{A}}_c = \bigcap_{u \in SS_c} \tilde{\mathcal{I}}_c(u),$$

where SS_c is the set of subsolution of (HJ) for $L - c$. These invariant sets contain the Mather set and have interesting dynamical properties, for instance $\tilde{\mathcal{A}}_c$ also is graph whose projection is laminated by global minimizers and it is chain recurrent. The Mañé set $\tilde{\mathcal{N}}_c$ is connected and chain transitive (see for instance [7]).

Using the duality between Lagrangian and Hamiltonian, via Legendre transform, we define the sets of Mather, Aubry and Mañé in the cotangent bundle, respectively by

$$\mathcal{M}_c^* = \mathcal{L}(\tilde{\mathcal{M}}_c), \mathcal{A}_c^* = \mathcal{L}(\tilde{\mathcal{A}}_c) \text{ e } \mathcal{N}_c^* = \mathcal{L}(\tilde{\mathcal{N}}_c).$$

One useful way to produce invariant Lipschitz Lagrangian graphs is to show that $\pi(\mathcal{A}_c^*) = M$. If this is the case, the Theorem 2.5 of [10] says that there exists an unique solution u of (HJ) for the Lagrangian $L - c$ which is $C^{1,1}$ and such that \mathcal{A}_c^* is the graph of $\eta_c + du$, for some η_c representative of cohomology class c .

3 Absorbing sets

If $u : M \rightarrow \mathbb{R}$ is a subsolution of (HJ) for $L - c$, we denote by $\tilde{\mathcal{I}}_c^+(u)$ the subset of TM defined as

$$\tilde{\mathcal{I}}_c^+(u) = \{(x, v) : \gamma_{(x,v)}|_{[0,+\infty)} \text{ is } (u, L - c, \alpha(c))\text{-calibrated}\},$$

where $\gamma_{(x,v)}$ is the curve defined on \mathbb{R} by

$$\gamma_{(x,v)}(t) = \pi \circ \varphi_t^L(x, v).$$

The *forward Mañé set* is defined by

$$\tilde{\mathcal{N}}_c^+ = \bigcup_{u \in SS_c} \tilde{\mathcal{I}}_c^+(u),$$

where SS_c is the set of critical subsolutions for the Lagrangian $L - c$. We define the *forward tiered Mañé set* as the union of all forward Mañé sets, i.e. the subset of TM given by

$$\mathcal{N}_+^T(L) = \bigcup_{c \in H^1(M)} \tilde{\mathcal{N}}_c^+.$$

Definition 3 We say that an invariant set $\Lambda \subset TM$ is an absorbing set if for all $(x, v) \in \mathcal{N}_+^T(L)$ we have

$$\omega(x, v) \subset \Lambda \Rightarrow (x, v) \in \Lambda$$

Definition 4 Let $\mathcal{G} \subset T^*M$ be an invariant Lipschitz Lagrangian graph. We say that \mathcal{G} is an absorbing graph if $\mathcal{L}^{-1}(\mathcal{G})$ is an absorbing set.

Lemma 5 If $(x, v) \in \tilde{\mathcal{N}}_c^+$, then the ω -limit set $\omega(x, v)$ is contained in $\tilde{\mathcal{A}}_c$.

Proof: In fact, let $(x, v) \in \tilde{\mathcal{N}}_c^+$ and $\gamma : \mathbb{R} \rightarrow M$ the projection of Euler-Lagrange solution $\gamma(t) = \pi \circ \varphi_t^L(x, v)$ curve such that $\gamma|_{[0, +\infty)}$ is $(\bar{u}, L - c, \alpha(c))$ -calibrated for some $\bar{u} \in SS_c$. This means that there exists a closed 1-form η_c with $[\eta_c] = c$ such that $H(x, \eta_c(x) + d_x \bar{u}) \leq \alpha(c)$ at almost every point and

$$\bar{u}(\gamma(t)) - \bar{u}(\gamma(t')) = A_{L-\eta_c+\alpha(c)}(\gamma|_{[t', t]}) \text{ for all } 0 < t' < t.$$

Let $(y, z) \in \omega(x, v)$, i.e. $(y, z) = \lim_{n \rightarrow \infty} (\gamma, \dot{\gamma})(t_n)$ with $t_n \rightarrow \infty$.

Let us consider $\sigma : \mathbb{R} \rightarrow M$ the projection of the Euler-Lagrange solution $\sigma(t) = \pi \circ \varphi_t^L(y, z)$. Given a Hamilton-Jacobi subsolution $u : M \rightarrow \mathbb{R}$ belonging to SS_c , there exists a closed 1-form ξ_c with $[\xi_c] = c$ and $H(x, \xi_c(x) + d_x u) \leq \alpha(c)$ at almost every point. It follows from ([9], Proposition 4.2.3) that u satisfies

$$u(\gamma(t)) - u(\gamma(t')) \leq A_{L-\xi_c+\alpha(c)}(\gamma|_{[t', t]}), \forall t' < t.$$

We can consider V a $C^\infty(M)$ function such that $\eta_c = \xi_c + dV$. Thus, if $s > 0$ and (t_k) and (t_m) are two subsequences of (t_n) such that $t_k - s > t_m + s$ and $t_k, t_m > s$, we have

$$\begin{aligned} & A_{L-\xi_c+\alpha(c)}(\sigma|_{[-s, s]}) + u(\sigma(-s)) - u(\sigma(s)) \\ &= \lim_{m, k} [A_{L-\xi_c+\alpha(c)}(\gamma|_{[t_m-s, t_m+s]}) + u(\gamma(t_k - s)) - u(\gamma(t_m + s))] \\ &\leq \lim_{m, k} [A_{L-\xi_c+\alpha(c)}(\gamma|_{[t_m-s, t_m+s]}) + A_{L-\xi_c+\alpha(c)}(\gamma|_{[t_m+s, t_k-s])}] \\ &= \lim_{m, k} A_{L-\xi_c+\alpha(c)}(\gamma|_{[t_m-s, t_k-s]}) \\ &= \lim_{m, k} [A_{L-\xi_c-dV+\alpha(c)}(\gamma|_{[t_m-s, t_k-s]}) + V(\gamma(t_k - s)) - V(\gamma(t_m - s))] \\ &= \lim_{m, k} A_{L-\eta_c+\alpha(c)}(\gamma|_{[t_m-s, t_k-s]}) + V(\gamma(-s)) - V(\gamma(-s)) \\ &= \lim_{m, k} \bar{u}(\gamma(t_k - s)) - \bar{u}(\gamma(t_m - s)) \\ &= \bar{u}(\gamma(-s)) - \bar{u}(\gamma(-s)) = 0. \end{aligned}$$

Therefore

$$A_{L-\xi_c+\alpha(c)}(\sigma|_{[-s,s]}) \leq u(\sigma(s)) - u(\sigma(-s)).$$

The opposite inequality holds because u is a Hamilton-Jacobi subsolution (see [9], Proposition 4.2.3). This shows that

$$(y, z) \in \bigcap_{u \in SS_c} \tilde{\mathcal{I}}_c(u) = \tilde{\mathcal{A}}_c.$$

■

Examples of absorbing graphs are the so-called *Schwartzman strictly ergodic graphs* (see [11]), i.e. invariant graphs Λ which support an invariant measure with full support. In fact, let μ the invariant measure supported in Λ with $\rho(\mu) = h$. If $\omega(x, p) \subset \Lambda$ for some $(x, p) \in \mathcal{L}(\tilde{\mathcal{I}}_c^+(L))$, then it follows from above lemma that $\omega(x, p) \subset \mathcal{A}_c^* \cap \Lambda$. In particular,

$$\omega(x, p) \subset \mathcal{L}(\tilde{\mathcal{M}}^h) \cap \mathcal{A}_c^*.$$

Since $\pi(\tilde{\mathcal{M}}^h) = M$, we have $\mathcal{A}_c^* = \mathcal{L}(\tilde{\mathcal{M}}^h) = \Lambda$. By the graph property, we conclude that $(x, p) \in \Lambda$.

Actually, the same argument can be used to prove that an invariant graph Λ such that all invariant probability measures with support contained in Λ have the same rotation vector h and the union of their supports equals Λ , also it is absorbing.

Proposition 6 *Let $\Lambda \subset TM$ be an invariant absorbing set. If $\tilde{\mathcal{A}}_c \subset \Lambda$ for some $c \in H^1(M; \mathbb{R})$, then Λ projects onto the whole manifold M .*

Proof: Let us consider a closed 1-form η_c representative of the cohomology class c . It follows from ([9], Theorem 4.9.3) that there exists a weak KAM of positive type u_+ for the Lagrangian $L - \eta_c$. Then u_+ is subsolution of (HJ) and given $x \in M$ we can find a C^1 curve $\gamma_x : [0, \infty) \rightarrow M$ with $\gamma_x(0) = x$, which is $(u_+, L - \eta_c, \alpha(c))$ -calibrated. This means that for all $0 < t' < t$ holds

$$u_+(\gamma_x(t)) - u_+(\gamma_x(t')) = \int_{t'}^t L(\gamma_x(s), \dot{\gamma}_x(s)) - \eta_c(\dot{\gamma}_x(s)) + \alpha(c) ds.$$

Therefore $(\gamma_x, \dot{\gamma}_x) \in \tilde{\mathcal{I}}_c^+(u_+) \subset \tilde{\mathcal{N}}_c^+$ and, by Lemma 5, we have that the ω -limit set of $(\gamma_x, \dot{\gamma}_x)$ is contained in Aubry set $\tilde{\mathcal{A}}_c$. Since Λ is an absorbing set that contains $\tilde{\mathcal{A}}_c$, we obtain $(x, \dot{\gamma}_x(0)) \in \Lambda$. ■

Lemma 7 *Let $c \in H^1(M; \mathbb{R})$ and $h \in H_1(M; \mathbb{R})$. We have $\widetilde{\mathcal{M}}^h \subset \widetilde{\mathcal{M}}_c$ if and only if $c \in \partial\beta(h)$.*

Proof: If $\widetilde{\mathcal{M}}^h \subset \widetilde{\mathcal{M}}_c$, then there exists a c -minimizing measure μ with $\rho(\mu) = h$. So

$$-\alpha(c) = \int_{TM} (L - \eta_c) d\mu = \int_{TM} L d\mu - \langle c, h \rangle = \beta(h) - \langle c, h \rangle,$$

where η_c is a representative of the cohomology class c . This show that $c \in \partial\beta(h)$.

Conversely, let μ minimizing measure with rotation vector h . If $c \in \partial\beta(h)$, then $\beta(h) = \langle c, h \rangle - \alpha(c)$. Therefore

$$-\alpha(c) = \beta(h) - \langle c, h \rangle = \int_{TM} L d\mu - \langle c, \rho(\mu) \rangle = \int_{TM} (L - \eta_c) d\mu.$$

This proves that μ is c -minimizing. ■

4 Proof of Theorems 1 and 2

In order to prove Theorems 1 and 2 we begin by proving the following lemma:

Lemma 8 *Let $\mathcal{G}_{\eta,u}$ be an invariant Lipschitz Lagrangian absorbing graph. Then $\mathcal{N}_{[\omega]}^* \cap \mathcal{G}_{\eta,u} \neq \emptyset$ if and only if $[\omega] = [\eta]$. Moreover, $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$.*

Proof: Since $\mathcal{G}_{\eta,u}$ is an invariant Lipschitz Lagrangian graph, it is contained in the Mañé's set $\mathcal{N}_{[\eta]}^*$. Therefore, if $(x_0, p) \in \mathcal{N}_{[\omega]}^* \cap \mathcal{G}_{\eta,u}$, then we have $(x_0, p) \in \mathcal{N}_{[\omega]}^* \cap \mathcal{N}_{[\eta]}^*$. By Lemma 5, the ω -limit set of points in Mañé set is contained in the Aubry set. Thus

$$\omega(x_0, p) \subset \mathcal{A}_{[\omega]}^* \cap \mathcal{A}_{[\eta]}^*.$$

This shows that the intersection $\mathcal{A}_{[\omega]}^* \cap \mathcal{A}_{[\eta]}^*$ is non-empty. Then, by a result of D. Massart (see [14], Proposition 6), α has a flat F containing $[\omega]$ and $[\eta]$. Let c be a cohomology class belonging to the relative interior of F . It follows from ([14], Proposition 6) that $\mathcal{A}_c^* \subset \mathcal{A}_{[\omega]}^* \cap \mathcal{A}_{[\eta]}^*$.

Let us consider a closed 1-form η_c , with $[\eta_c] = c$. It follows from ([9], Theorem 4.9.3) that there exists a weak KAM of positive type u_+ for the Lagrangian $L - \eta_c$. Then u_+ is subsolution of (HJ) and given $x \in M$, we can find a C^1 curve $\gamma_x : [0, \infty) \rightarrow M$ with $\gamma_x(0) = x$, which is $(u_+, L - \eta_c, \alpha(c))$ -calibrated. This means that for all $0 < t' < t$ holds

$$u_+(\gamma_x(t)) - u_+(\gamma_x(t')) = \int_{t'}^t L(\gamma_x(s), \dot{\gamma}_x(s)) - \eta_c(\dot{\gamma}_x(s)) + \alpha(c) ds.$$

Therefore $(\gamma_x, \dot{\gamma}_x) \in \widetilde{\mathcal{N}}_c^+$ and, by Lemma 5, we have that the ω -limit set of $(\gamma_x, \dot{\gamma}_x)$ is contained in Aubry set $\widetilde{\mathcal{A}}_c$. Recall that by invariance of $\mathcal{G}_{\eta,u}$ we have $\mathcal{G}_{\eta,u} = \mathcal{L}(\widetilde{\mathcal{I}}_\eta(u))$ and the Aubry set $\mathcal{A}_{[\eta]}^*$ is contained in $\mathcal{G}_{\eta,u}$. Since $\mathcal{A}_c^* \subset \mathcal{A}_{[\eta]}^* \subset \mathcal{G}_{\eta,u}$, we conclude the ω -limit set of $(\gamma_x, \dot{\gamma}_x)$ is contained in $\mathcal{G}_{\eta,u}$. Moreover, it follows from $\mathcal{G}_{\eta,u}$ being an absorbing graph, that the Hamiltonian orbit $\mathcal{L}(\gamma_x, \dot{\gamma}_x)$ is entirely contained in $\mathcal{G}_{\eta,u}$. As a consequence, given $T > 0$, we have

$$\mathcal{L}(\gamma_x(-T), \dot{\gamma}_x(-T)) \in \mathcal{G}_{\eta,u}.$$

If we let $z = \gamma_x(-T) \in M$, since u_+ is weak KAM, there exists a C^1 curve $\gamma_z : [0, \infty) \rightarrow M$ which is $(u_+, L - \eta_c, \alpha(c))$ -calibrated with $\gamma_z(0) = z$. Similarly as above, the ω -limit set of $\mathcal{L}(\gamma_z, \dot{\gamma}_z)$ is contained in $\mathcal{G}_{\eta,u}$. Thus $\mathcal{L}(\gamma_z(0), \dot{\gamma}_z(0)) \in \mathcal{G}_{\eta,u}$ and, since $\gamma_z(0) = \gamma_x(-T)$, we have $\dot{\gamma}_z(0) = \dot{\gamma}_x(-T)$. It follows from the uniqueness of solutions that $\gamma_z(t) = \gamma_x(t - T)$. This shows that $\gamma_x|_{[-T, \infty)}$ is $(u_+, L - c, \alpha(c))$ -calibrated and, by ([9], Lemma 4.13.1), u_+ is differentiable at x with

$$d_x u_+ = \frac{\partial L}{\partial v}(x, \dot{\gamma}_x(0)) - \eta_c(\dot{\gamma}_x(0)).$$

Now let us consider another weak KAM v_+ for the Lagrangian $L - \eta_c$. There exists a C^1 curve $\delta_x : [0, \infty) \rightarrow M$ with $\delta_x(0) = x$, which is $(v_+, L - \eta_c, \alpha(c))$ -calibrated. Similarly, we conclude that

$$d_x v_+ = \frac{\partial L}{\partial v}(x, \dot{\delta}_x(0)) - \eta_c(\dot{\delta}_x(0)).$$

Moreover, the points $\mathcal{L}(\gamma_x(0), \dot{\gamma}_x(0))$ and $\mathcal{L}(\delta_x(0), \dot{\delta}_x(0))$ belong to the graph $\mathcal{G}_{\eta,u}$ with $\gamma_x(0) = \delta_x(0) = x$. Thus $\dot{\gamma}_x(0) = \dot{\delta}_x(0)$ and we conclude that $d_x u_+ = d_x v_+$. Since this equality holds for all $x \in M$ and M is connected, we have that u_+ differ of v_+ by a constant. By arbitrariness of the two weak KAM, we conclude that any two weak KAM for the Lagrangian $L - \eta_c$ differ by a constant. It follows from ([10], Proposition 4.4) that $\mathcal{A}_c^* = \mathcal{N}_c^*$. On the other hand, since u_+ is differentiable everywhere in M , by ([9], Lemma 4.13.1), we have

$$H(x, \eta_c(x) + d_x u_+) = \alpha(c), \quad (2)$$

This means that the graph $\mathcal{G}_{\eta_c, u_+}$ is invariant, so $\mathcal{G}_{\eta_c, u_+} \subset \mathcal{N}_c^* = \mathcal{A}_c^*$. By the graph property, we have $\mathcal{A}_c^* = \mathcal{N}_c^* = \mathcal{G}_{\eta_c, u_+}$. As a consequence of $\mathcal{A}_c^* \subset \mathcal{A}_{[\omega]}^* \cap \mathcal{A}_{[\eta]}^*$, we obtain

$$\mathcal{A}_c^* = \mathcal{A}_{[\omega]}^* = \mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}.$$

We also have that the projected Aubry set $\mathcal{A}_{[\omega]}$ is the whole manifold M , so there exists a function $v \in C^{1,1}$ such that $\mathcal{A}_{[\omega]}^* = \mathcal{G}_{\omega,v}$. Therefore

$$\eta(x) + d_x u = \omega(x) + d_x v, \forall x \in M \Rightarrow (\eta - \omega)(x) = d_x(v - u),$$

which implies $[\omega] = [\eta]$. ■

We can now prove the main result stated in Introduction.

Proof: (of Theorem 1) Suppose that $\mathcal{G}_{\eta,u}$ is an absorbing graph. By the Lemma 8, $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$. We need to show that β is differentiable on $\partial\alpha([\eta])$. For a given $h \in \partial\alpha([\eta])$, it suffices to show that $\partial\beta(h) = \{[\eta]\}$. Let us assume that $[\xi] \in \partial\beta(h)$, so we have $h \in \partial\alpha([\eta]) \cap \partial\alpha([\xi])$. It follows from Lemma 7 that

$$\widetilde{\mathcal{M}}^h \subset \widetilde{\mathcal{A}}_{[\eta]} \cap \widetilde{\mathcal{A}}_{[\xi]}.$$

This implies that $\mathcal{A}_{[\eta]}^* \cap \mathcal{A}_{[\xi]}^* \neq \emptyset$. Moreover, since $\mathcal{A}_{[\eta]}^* \subset \mathcal{G}_{\eta,u}$, by Lemma 8 we have $[\xi] = [\eta]$. Hence β is differentiable at h .

Conversely, suppose that β is differentiable at all homology class $h \in \partial\alpha([\eta])$ and $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$. Let $(x, p) \in \mathcal{N}_+^T(L)$ such that $\omega(x, p) \subset \mathcal{G}_{\eta,u}$ and let us consider $c \in H^1(M; \mathbb{R})$ and $v \in SS_c$ such that $(x, p) \in \mathcal{L}(\widetilde{\mathcal{I}}_c^+(v))$. It follows from Lemma 5 that $\omega(x, p) \subset \mathcal{A}_c^*$. Thus

$$\omega(x, p) \subset \mathcal{A}_c^* \cap \mathcal{G}_{\eta,u} = \mathcal{A}_c^* \cap \mathcal{A}_{[\eta]}^*.$$

This implies that $\mathcal{A}_c^* \cap \mathcal{A}_{[\eta]}^* \neq \emptyset$ and, by ([14], Proposition 6), there exists a flat F of α such that c and $[\eta]$ belong to F . As a consequence, there exists $h \in H_1(M; \mathbb{R})$ such that $[\eta], c \in \partial\beta(h)$. Since β is differentiable at all $h \in \partial\alpha([\eta])$, we obtain $[\eta] = c$. In particular, $(x, p) \in \mathcal{L}(\widetilde{\mathcal{I}}_{[\eta]}^+(v))$. Since $\mathcal{A}_{[\eta]} = M$, there exists $\xi \in T_x M$ such that $(x, \xi) \in \widetilde{\mathcal{A}}_{[\eta]}$. Then v is differentiable at x and $\frac{\partial L}{\partial v}(x, \xi) - \eta = d_x v$ (see [10], Theorem 2.5). Moreover, if $\gamma = \pi \circ \varphi_t^L(\mathcal{L}^{-1}(x, p))$ is the curve such that $\gamma|_{[0, +\infty)}$ is $(u, L - [\eta], \alpha([\eta]))$ -calibrated, then

$$v(\gamma(t)) - v(\gamma(0)) = A_{L-\eta+\alpha([\eta])}(\gamma|_{[0,t]}) \text{ for all } t > 0.$$

Dividing by t , and letting $t \rightarrow 0^+$, we get

$$d_x v(\dot{\gamma}(0)) = L(x, \dot{\gamma}(0)) - \eta(\dot{\gamma}(0)) + \alpha([\eta]).$$

By Fenchel inequality, this can happen if and only if $\frac{\partial L}{\partial v}(x, \dot{\gamma}(0)) - \eta = d_x v$. Therefore $\dot{\gamma}(0) = \xi$ and $(x, p) \in \mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$. ■

Other examples of absorbing graphs are graphs contained in neighborhoods foliated by invariant Lipschitz Lagrangian graphs. We say that an open \mathcal{V} in T^*M is foliated by invariant Lipschitz Lagrangian graphs if each $(x, p) \in \mathcal{V}$ belongs to a unique invariant Lipschitz Lagrangian graph $\mathcal{G} \subset \mathcal{V}$.

Definition 9 *We say that a Tonelli Hamiltonian H is locally Lipschitz integrable on an invariant Lipschitz Lagrangian graph $\mathcal{G}_{\eta,u}$ if there exists a neighborhood $\mathcal{V} \subset T^*M$ of $\mathcal{G}_{\eta,u}$ foliated by invariant Lipschitz Lagrangian graphs.*

Now we can to prove the Theorem 2:

Proof: (of Theorem 2) Let \mathcal{V} be a neighborhood of $\mathcal{G}_{\eta,u}$ foliated by invariant Lipschitz Lagrangian graphs. Note that an invariant Lipschitz Lagrangian graph \mathcal{G} contained in \mathcal{V} is absorbing. Since $\mathcal{G}_{\eta,u} \subset \mathcal{V}$, by Theorem 1,

$$\mathcal{G}_{\eta,u} = \mathcal{A}_{[\eta]}^* = \mathcal{N}_{[\eta]}^*.$$

We can use the upper semicontinuity of the Mañé set (see for instance [1], Proposition 13]) to deduce that there exists $U_0 \subset H^1(M; \mathbb{R})$, an open neighborhood of $[\eta]$, such that $\mathcal{N}_c^* \subset \mathcal{V}$ for all $c \in U_0$. Given

$$h \in V = \bigcup_{c \in U_0} \partial\alpha(c),$$

let us suppose that $h \in \partial\alpha(c_0)$ for some $c_0 \in U_0$. So $\mathcal{N}_{c_0}^* \subset \mathcal{V}$ and, by connectedness of Mañé set, $\mathcal{N}_{c_0}^*$ is contained in some invariant Lipschitz Lagrangian $\mathcal{G}_{\omega,v} \subset \mathcal{V}$. Since $\mathcal{G}_{\omega,v}$ is absorbing, it follows from Lemma 8 that $c_0 = [\omega]$. Moreover, by the Theorem 1, β is differentiable at $h \in \partial\alpha(c_0)$. ■

5 Dynamical properties on a neighborhood of graphs and differentiability of β

The tiered Mañé set, denoted by $\mathcal{N}^T(L)$ was introduced by M-C. Arnaud in [1]. It is defined as the union of Mañé sets $\tilde{\mathcal{N}}_c$ for all cohomology class $c \in H^1(M; \mathbb{R})$, i.e.

$$\mathcal{N}^T(L) = \bigcup_{c \in H^1(M)} \tilde{\mathcal{N}}_c.$$

Arnaud also defines the dual tiered Mañé $\mathcal{N}_*^T(L)$ as $\mathcal{L}(\mathcal{N}^T(L)) \subset T^*M$ and proves, in the work *A particular minimization property implies C^0 -integrability* (see [2]), that

the dual tiered Mañé $\mathcal{N}_*^T(L)$ is whole cotangent bundle T^*M if and only if T^*M is foliated by invariant Lipschitz Lagrangian graph. In this section we are interested in the differentiability of β when there exists a neighborhood \mathcal{V} of an invariant Lipschitz Lagrangian \mathcal{G} contained in $\mathcal{N}_*^T(L)$. We study the relation of this hypothesis with the existence of absorbing graphs contained in the neighborhood \mathcal{V} .

Lemma 10 *Let us assume that for $[\eta] \in H^1(M; \mathbb{R})$ there exists a neighborhood \mathcal{V} of $\mathcal{A}_{[\eta]}^*$ in T^*M such that $\mathcal{V} \subset \mathcal{N}_*^T(L)$. Hence if for $c \in H^1(M; \mathbb{R})$, $\mathcal{A}_c^* \cap \mathcal{V}$ is non-empty, then $\mathcal{A}_{[\eta]} \subset \mathcal{A}_c$.*

Proof: Let η_c be a representative of the cohomology class c . Let us consider $(x_0, p_0) \in \mathcal{A}_c^* \cap \mathcal{V}$ and $\delta > 0$ such that the ball $B_\delta(x_0, p_0) \subset T^*M$ centered at (x_0, p_0) and radius δ is contained in \mathcal{V} . By ([2], Proposition 10), we can find $T_0(c) > 0$ such that for all $T > T_0(c)$ and for every Tonelli minimizing curve $\gamma : [0, T] \rightarrow M$ with $\gamma(0) = \gamma(T)$ for the Lagrangian $L - \eta_c$, we have $d(\mathcal{L}(x_0, \dot{\gamma}(0)), (x_0, p_0)) < \delta$, where d is the distance in T^*M .

Since $x_0 \in \mathcal{A}_c$, one of characterizations of Aubry sets (see for instance [10], Theorem 2.1 (4)) ensures the existence of a sequence of Tonelli minimizing curves $\gamma_n : [0, T_n] \rightarrow M$ for the Lagrangian $L - c$ with $\gamma_n(0) = \gamma_n(T_n) = x_0$ such that $T_n \rightarrow \infty$ and $A_{L-c+\alpha(c)}(\gamma_n) \rightarrow 0$. Therefore, if $T_n > T_0(c)$, we have

$$d(\mathcal{L}(x_0, \dot{\gamma}_n(0)), (x_0, p_0)) < \delta, \text{ hence } \mathcal{L}(x_0, \dot{\gamma}_n(0)) \in \mathcal{V} \subset \mathcal{N}_*^T(L).$$

In particular, $(\gamma_n, \dot{\gamma}_n)$ is a periodic orbit contained in some Mañé set $\tilde{\mathcal{N}}_{[\lambda_n]}$ for some closed 1-form λ_n . Therefore $(\gamma_n, \dot{\gamma}_n)$ supports a measure μ_n which is $[\lambda_n]$ -minimizing.

Now we will show that $\mathcal{A}_{[\eta]} \subset \mathcal{A}_c$. Indeed, let us consider $y \in \mathcal{A}_{[\eta]}$ and $q \in T_y^*M$ such that $(y, q) \in \mathcal{A}_{[\eta]}^*$. Let $\epsilon > 0$ such that $B_\epsilon(y, q) \subset \mathcal{V}$. Again by ([2], Proposition 10), we can find $T_0([\eta]) > 0$ such that for all $T > T_0([\eta])$ and for every Tonelli minimizing curve $\zeta : [0, T] \rightarrow M$ with $\zeta(0) = \zeta(T)$ for the Lagrangian $L - [\eta]$, we have $d(\mathcal{L}(y, \dot{\zeta}(0)), (y, q)) < \epsilon$.

Take n sufficiently large such that $T_n > \max\{T_0(c), T_0([\eta])\}$. Because of Tonelli Theorem, we know that for each T_n , there exists a Tonelli minimizing curve $\Gamma_n : [0, T_n] \rightarrow M$ with $\Gamma_n(0) = \Gamma_n(T_n) = y$ for the Lagrangian $L - \eta$ which is homologous to γ_n . Since $T_n > T_0([\eta])$, we have $d(\mathcal{L}(y, \dot{\Gamma}_n(0)), (y, q)) < \epsilon$ which implies

$$\mathcal{L}(y, \dot{\Gamma}_n(0)) \in B_\epsilon(y, q) \subset \mathcal{V}.$$

As a consequence, $(\Gamma_n, \dot{\Gamma}_n)$ is a periodic orbit which supports a measure ν_n which is $\bar{\lambda}_n$ -minimizing for some closed 1-form $\bar{\lambda}_n$. However, we have

$$\rho(\mu_n) = \frac{1}{T_n} [\gamma_n] = \frac{1}{T_n} [\Gamma_n] = \rho(\nu_n).$$

Thus,

$$\begin{aligned} A_{L-\lambda_n}(\mu_n) &= A_L(\mu_n) - \langle \rho(\mu_n), \lambda_n \rangle \\ &= \beta(\rho(\mu_n)) - \langle \rho(\nu_n), \lambda_n \rangle \\ &= A_{L-\lambda_n}(\nu_n). \end{aligned}$$

Or,

$$\int_0^{T_n} L(\gamma_n, \dot{\gamma}_n) - \lambda_n(\dot{\gamma}_n) dt = \int_0^{T_n} L(\Gamma_n, \dot{\Gamma}_n) - \lambda_n(\dot{\Gamma}_n) dt.$$

Therefore,

$$\begin{aligned} A_{L-\eta_c+\alpha(c)}(\Gamma_n) &= A_{L-\lambda_n+(\lambda_n-\eta_c)+\alpha(c)}(\Gamma_n) = A_{L-\lambda_n}(\Gamma_n) + [\lambda_n - \eta_c][\Gamma_n] + \alpha(c)T_n \\ &= A_{L-\lambda_n}(\gamma_n) + [\lambda_n - \eta_c][\gamma_n] + \alpha(c)T_n = A_{L-\eta_c+\alpha(c)}(\gamma_n) \rightarrow 0. \end{aligned}$$

This shows that $y \in \mathcal{A}_c$ and we obtain $\mathcal{A}_{[\eta]} \subset \mathcal{A}_c$. ■

The following corollary relates a neighborhood contained in the dual tiered Mañé to the existence of absorbing graphs for each cohomology class belonging to an open subset of $H^1(M; \mathbb{R})$.

Corollary 11 *Let \mathcal{V} be a neighborhood of an invariant Lipschitz Lagrangian graph $\mathcal{G}_{\eta,u}$ such that $\mathcal{G}_{\eta,u} = \mathcal{A}_{[\eta]}^*$. Let us assume that \mathcal{V} is contained in the dual tiered Mañé $\mathcal{N}_*^T(L)$. Then there exists a neighborhood $U_0 \subset H^1(M; \mathbb{R})$ of $[\eta]$ such that for each $c \in U_0$, there exists an absorbing graph $\mathcal{G}_{\eta_c, u_c}$ with $[\eta_c] = c$ and $u_c \in C^{1,1}$. Hence β is differentiable at any point of $V = \bigcup_{c \in U_0} \partial\alpha(c)$.*

Proof: We can use the upper semicontinuity of the Mañé set (see for instance [1], Proposition 13) to deduce that there exists $U_0 \subset H^1(M; \mathbb{R})$, a neighborhood of $[\eta]$, such that $\mathcal{N}_c^* \subset \mathcal{V}$ for all $c \in U_0$. In particular, for all $c \in U_0$ we have $\mathcal{A}_c^* \subset \mathcal{V}$ and, by the Lemma 10, we conclude that $\mathcal{A}_c = \mathcal{A}_{[\eta]}$. Moreover, since $\mathcal{G}_{\eta,u} = \mathcal{A}_{[\eta]}^*$, we have $\mathcal{A}_c = M$. Therefore, there exists a closed 1-form η_c with $[\eta_c] = c$ and a function $u_c \in C^{1,1}$ such that $\mathcal{A}_c^* = \mathcal{G}_{\eta_c, u_c}$. It remains to show that $\mathcal{G}_{\eta_c, u_c}$ is absorbing. In fact, if $(x, p) \in \mathcal{L}(\mathcal{N}_+^T(L))$ and

$$\omega(x, p) \subset \mathcal{G}_{\eta_c, u_c} = \mathcal{A}_c^*,$$

then the curve $\gamma(t) = \pi \circ \varphi_t^H(x, p)$, the projection of the Hamiltonian flow φ_t^H , intersects \mathcal{V} for some time $\tau > 0$. In particular $\varphi_\tau^H(x, p)$ belongs to some Mañé set $\mathcal{N}_{\bar{c}}^*$. This implies that $\omega(x, p) \subset \mathcal{A}_{\bar{c}}^*$, so $\mathcal{A}_{\bar{c}}^* \cap \mathcal{A}_c^* \neq \emptyset$. The Lemma 10 implies that $\mathcal{A}_{\bar{c}} = M$ and, as a consequence, there exist $v \in C^{1,1}$ and $\eta_{\bar{c}}$ a representative of the cohomology class \bar{c} such that $\mathcal{A}_{\bar{c}}^* = \mathcal{G}_{\eta_{\bar{c}}, v}$.

By Proposition 6 of [14], c and \bar{c} belong to the same flat F of α . Moreover, if c_0 is in the relative interior of F , then $\mathcal{A}_{c_0}^* \subset \mathcal{A}_c^* \cap \mathcal{A}_{\bar{c}}^*$. It follows from Lemma 10 that $\mathcal{A}_{c_0} = M$ and, as a consequence, there exist a closed 1-form η_{c_0} with $[\eta_{c_0}] = c_0$ and a function $u_0 \in C^{1,1}$ such that $\mathcal{A}_{c_0}^* = \mathcal{G}_{\eta_{c_0}, u_0}$. This shows that

$$\mathcal{G}_{\eta_{c_0}, u_0} = \mathcal{G}_{\eta_c, u} = \mathcal{G}_{\eta_{\bar{c}}, v},$$

and, by the graph property, $\mathcal{N}_{\bar{c}}^* = \mathcal{G}_{\eta_{\bar{c}}, v} = \mathcal{G}_{\eta_c, u}$. This shows that $\varphi_t^H(x, p)$ belongs to $\mathcal{G}_{\eta_c, u}$ for all $t \in \mathbb{R}$. In particular, (x, p) belongs to $\mathcal{G}_{\eta_c, u}$ and we conclude that $\mathcal{G}_{\eta_c, u}$ is an absorbing graph.

The conclusion that β is differentiable at any point of $V = \bigcup_{c \in U_0} \partial\alpha(c)$ follows from Theorem 1. ■

The union of the invariant absorbing graphs obtained in the previous corollary may or may not be an open subset of T^*M . However, this is the case, with the additional hypothesis $\dim H^1(M; \mathbb{R}) = \dim M$, as shown in the following theorem.

Theorem 12 *Suppose that $\dim H^1(M; \mathbb{R}) = \dim M$. Let $U_0 \subset H^1(M, \mathbb{R})$ be a neighborhood of $[\eta]$ such that for each $c \in U_0$, there exists an invariant Lipschitz Lagrangian absorbing graph $\mathcal{G}_{\eta_c, u_c}$ with $[\eta_c] = c$ and $u_c \in C^{1,1}$. Then the Hamiltonian is locally Lipschitz integrable on $\mathcal{G}_{\eta, u}$, where $u = u_{[\eta]}$.*

Proof: Let us define the map

$$\begin{aligned} F : M \times U_0 &\rightarrow T^*M \\ (x, c) &\rightarrow (x, \eta_c(x) + d_x u_c) \end{aligned} \tag{3}$$

where the closed 1-form η_c is a representative of c and $u_c \in C^{1,1}$ such that $\mathcal{G}_{\eta_c, u_c}$ is an invariant absorbing graph. In particular, by Theorem 1, we have $\mathcal{A}_c^* = \mathcal{G}_{\eta_c, u_c}$. This map is injective because the absorbing graphs are disjoint. Moreover, F is also continuous. In fact, let $(x_n, c_n) \rightarrow (x_0, c)$ and consider its associated graphs $\mathcal{A}_{c_n}^* = \mathcal{G}_{\eta_{c_n}, u_{c_n}}^*$. Let us consider the sequence of Lipschitz functions $\lambda_n = \eta_{c_n} + du_{c_n}$. Note that, by the graph property, we have $\mathcal{A}_{c_n}^* = \mathcal{N}_{c_n}^*$. Moreover, by upper semicontinuity of the Mañé set (see [1], Proposition 13), we conclude that for n sufficiently large, the sequence

$(x_n, \lambda_n(x_n)) \in \mathcal{N}_{c_n}^*$ remains in a compact set. We can conclude that – up to selecting a subsequence – $(x_n, \lambda_n(x_n))$ converges to some point that belongs to \mathcal{N}_c^* . Moreover, since $\mathcal{N}_{c_n}^* = \mathcal{A}_{c_n}^*$ and $x_n \rightarrow x_0$, by the graph property, we conclude that

$$(x_n, \lambda_n(x_n)) \rightarrow (x_0, \eta_c(x_0) + d_{x_0}u_c) \Leftrightarrow F(x_n, c_n) \rightarrow F(x_0, c).$$

The result follows from Invariance Domain Theorem for manifolds. In fact,

$$\dim(M \times U) = 2n = \dim T^*M$$

and F is continuous and injective. Therefore $F(M \times U)$ is open and $\mathcal{G}_{\eta, u} \subset F(M \times U)$, that is, the Hamiltonian is locally Lipschitz integrable on $\mathcal{G}_{\eta, u}$. ■

As an immediate consequence of Corollary 11 and Theorem 12, we present a local version of Arnaud's Theorem ([2], Theorem 1).

Corollary 13 *Suppose that $\dim H^1(M; \mathbb{R}) = \dim M$. Let \mathcal{V} be a neighborhood of an invariant Lipschitz Lagrangian $\mathcal{G}_{\eta, u}$ such that $\mathcal{G}_{\eta, u} = \mathcal{A}_{[\eta]}^*$ and \mathcal{V} is contained in the dual tiered Mañé $\mathcal{N}_*^T(L)$. Then the Hamiltonian is locally Lipschitz integrable on $\mathcal{G}_{\eta, u}$.*

6 The Case $M = \mathbb{T}^2$

We say that a homology $h \in H_1(\mathbb{T}^2; \mathbb{R})$ is *rational* if there exists $\lambda > 0$ such that $\lambda h \in i_* H_1(\mathbb{T}^2; \mathbb{Z})$, where $i_* : H_1(\mathbb{T}^2; \mathbb{Z}) \rightarrow H_1(\mathbb{T}^2; \mathbb{R})$ is the natural map. Since the manifold treated in this section is the torus \mathbb{T}^2 , we can say that a homology $h \in H_1(\mathbb{T}^2; \mathbb{R})$ is *irrational* if it is not rational. For general manifolds, there is the concept of *k-irrationality* (see for instance [14]).

Note that if two cohomology classes lie in the relative interior of a flat of α , by [17] their Mather sets coincide. Then $\widetilde{\mathcal{M}}(\partial\beta(h))$ denotes the Mather sets of all the cohomologies in the relative interior of $\partial\beta(h)$. A homology class h is said to be singular if its Legendre transform $\partial\beta(h)$ is a singular flat, i.e. its Mather set $\widetilde{\mathcal{M}}(\partial\beta(h))$ contains fixed points.

One natural question in the present context is if the set $V = \bigcup_{c \in U_0} \partial\alpha(c)$ obtained in Theorem 2 is an open of $H_1(M; \mathbb{R})$. If this the case, by convexity β , we obtain that it is of class C^1 in V . In the case of $M = \mathbb{T}^2$, a positive answer to this question is given in the next proposition.

Proposition 14 *Suppose $M = \mathbb{T}^2$. Let $c_0 \in H^1(\mathbb{T}^2; \mathbb{R})$ and U_0 be a neighborhood of c_0 in $H^1(\mathbb{T}^2; \mathbb{R})$ such that β is differentiable at any point of $V = \bigcup_{c \in U_0} \partial\alpha(c)$. Then α is C^1 in U_0 . In particular, V is an open set of $H_1(\mathbb{T}^2; \mathbb{R})$ and β is of class C^1 in V .*

Proof: Let $c \in U_0$ be a cohomology class and let h be a rational homology class belonging to $\partial\alpha(c)$. Suppose by contradiction that $\partial\alpha(c) \neq \{h\}$. Recall that, in case $M = \mathbb{T}^2$, all flats of β are radial (see [5]). Therefore, exchanging, if necessary, h and a nonzero extremal point of $\partial\alpha(c)$, which exists because $\partial\alpha(c)$ has two extremal points which also are rational, we can assume

$$\partial\alpha(c) = [\lambda h, h] = \{th : t \in [\lambda, 1]\}$$

for some $\lambda < 1$. Let us consider a sequence $t_n > 1$ such that $t_n \rightarrow 1$. We will show that, for n sufficiently large, β is differentiable at $t_n h$. In fact, there exists a sequence $c_n \in H^1(\mathbb{T}^2; \mathbb{R})$ such that $c_n \in \partial\beta(t_n h)$. Thus

$$\beta(t_n h) = \langle t_n h, c_n \rangle - \alpha(c_n).$$

The sequence c_n has subsequence bounded. Otherwise, let us consider a subsequence of $\frac{c_n}{\|c_n\|}$ convergent, i.e. $\frac{c_{n_k}}{\|c_{n_k}\|} \rightarrow a$, where $\|\cdot\|$ denotes a norm on $H^1(\mathbb{T}^2; \mathbb{R})$ (see for instance [9], Section 4.10). So we have $\|c_{n_k}\| \rightarrow \infty$ and

$$\frac{\alpha(c_{n_k})}{\|c_{n_k}\|} = \left\langle t_{n_k} h, \frac{c_{n_k}}{\|c_{n_k}\|} \right\rangle - \frac{\beta(t_{n_k} h)}{\|c_{n_k}\|} \rightarrow \langle h, a \rangle,$$

which contradicts the superlinearity of α . Then we can assume, extracting a subsequence if necessary, $c_n \rightarrow \tilde{c}$. Thence

$$\beta(h) = \lim \beta(t_n h) = \lim \langle t_n h, c_n \rangle - \alpha(c_n) = \langle h, \tilde{c} \rangle - \alpha(\tilde{c}).$$

This shows that $\tilde{c} \in \partial\beta(h)$. Since β is differentiable at h , we conclude that $\tilde{c} = c$ and $c_n \in U_0$ for n sufficiently large. Therefore β is differentiable at $t_n h$.

Observe that $t_n h$ is non-singular for all $n \in \mathbb{N}$. Otherwise, there exists a fixed point, which comprise the support of a minimizing measure μ_0 , in $\widetilde{\mathcal{M}}(\partial\beta(t_n h)) = \widetilde{\mathcal{M}}_{c_n}$. Since $\rho(\mu_0) = 0$, the set $[0, t_n h]$ is contained in a flat of β and the maximal flat $[\lambda h, h]$ may be extended because $t_n > 1$.

Therefore we can apply ([16], Corollary 1) to deduce that \mathbb{T}^2 is foliated by periodic orbits and $\mathcal{M}_{c_n} = \mathcal{A}_{c_n} = \mathbb{T}^2$. By ([8], Proposition 2.1) we can consider a point $(x, v) \in T\mathbb{T}^2$ such that the orbit $\varphi_t(x, v)$ is periodic with period T and which comprise the support of a minimizing measure with rotation vector λh . If (x, v) is a fixed point, take $T = +\infty$. Since $x \in \mathcal{A}_{c_n}$, there exists a only v_n such that $(x, v_n) \in \widetilde{\mathcal{A}}_{c_n}$ is a periodic point with period T_n . By semicontinuity of the Aubry set, (x, v_n) converges to some point (x, w) of $\widetilde{\mathcal{A}}_c$. By the graph property, $w = v$.

We now prove that for all $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $T_{n_0} \geq T - \delta$. In fact, if some $\delta > 0$ is such that $0 < T_n < T - \delta$ for all $n \in \mathbb{N}$, extracting a subsequence if necessary, $T_n \rightarrow S < T$ and $\varphi_{T_n}(x, v_n) \rightarrow \varphi_S(x, v) = (x, v)$ which contradicts the minimality of period of the orbit of (x, v) . Let us consider $h_0 \in H_1(\mathbb{T}^2; \mathbb{Z})$ such that the probability measure carried by the orbit $\varphi_t(x, v)$ have homology $\frac{1}{T}h_0 = \lambda h$ and the probability measure carried by the orbit $\varphi_t(x, v_n)$ have homology $\frac{1}{T_n}h_0 = t_n h$. Given $\delta = T(1 - \lambda)$, there exists $n_0 \in \mathbb{N}$ such that $T_{n_0} \geq T - T(1 - \lambda) = T\lambda$. Then, since $h_0 = T\lambda h$ and $h_0 = T_{n_0}t_{n_0}h$, we have $\frac{T\lambda}{T_{n_0}} = t_{n_0} \leq 1$ which is a contradiction.

Now suppose that h be an irrational homology class belongs to $\partial\alpha(c)$. Therefore any probability measure h -minimizing is supported on a lamination of the torus without closed leaves. Moreover, this measure is uniquely ergodic. In particular, h is not contained in any non-trivial flat of β . \blacksquare

As a consequence of the above Proposition, we present a local version of ([14], Theorem 3).

Theorem 15 *Suppose that $M = \mathbb{T}^2$ and let $[\eta]$ be a cohomology class. Then the following statements are equivalent:*

- (1) *There exists $u \in C^{1,1}(M)$ and a neighborhood \mathcal{V} of $\mathcal{G}_{\eta,u}$ in T^*M such that $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta,u}$ and $\mathcal{V} \subset \tilde{\mathcal{N}}^T(L)$.*
- (2) *β is differentiable at any point of $V = \bigcup_{c \in U_0} \partial\alpha(c)$ for some neighborhood $U_0 \subset H^1(M; \mathbb{R})$ of $[\eta]$.*
- (3) *There exists $u \in C^{1,1}(M)$ such that the Hamiltonian is locally Lipschitz integrable on $\mathcal{G}_{\eta,u}$.*

Proof: (1) \Rightarrow (2) It follows immediately of Corollary 11.

(2) \Rightarrow (3) By Proposition 14 V is open in $H_1(\mathbb{T}^2; \mathbb{R})$ and the Legendre transform is a homeomorphism between V and U_0 . For each $h \in V$, let us consider $c_h = \partial\beta(h)$. Then the set of the classes c_h with h rational and non-singular is dense in U_0 . In fact, the set of the classes c_h with h rational (see [14], Lemma 7) is dense in $H^1(\mathbb{T}^2; \mathbb{R})$. Now note that zero is the only (possibly) singular class belongs to $\partial\alpha(c)$ such that $c \in U_0$, because if a non-zero class h is singular belongs to $\partial\alpha(c)$ such that $c \in U_0$, then there is a fixed point in the Mather set of c . Thus $\partial\alpha(c)$ contains the homology of the Dirac measure on the fixed point. This contradicts the Proposition 14, which says that α is C^1 in U_0 .

It follows from ([16], Corollary 1) that $\mathcal{A}_{c_h} = \mathbb{T}^2$ is foliated by periodic orbits. By semicontinuity of Aubry set, we have that $\mathcal{A}_c = \mathbb{T}^2$ for all $c \in U_0$ and that there exist

η_c a representative of class c and $u_c \in C^{1,1}(\mathbb{T}^2)$ such that $\mathcal{A}_c^* = \mathcal{G}_{\eta_c, u_c}$. In particular, $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta, u}$ for some $u \in C^{1,1}$. Moreover, these graphs are absorbing. Indeed, this follows from assumption that β is differentiable at any point $V = \bigcup_{c \in U_0} \partial\alpha(c)$ and from converse of Theorem 1. Therefore, since $\dim H_1(\mathbb{T}^2; \mathbb{R}) = \dim \mathbb{R}^2$, by the Theorem 12, we obtain a neighborhood of $\mathcal{G}_{\eta, u}$ foliated by invariant Lipschitz Lagrangian graphs.

(3) \Rightarrow (1) All point of \mathcal{V} belongs to some invariant Lipschitz Lagrangian graphs. In particular, belongs to some Mañé set, so $\mathcal{V} \subset \tilde{\mathcal{N}}^T(L)$. Since graphs of a foliation are absorbing, by Theorem 1 we have $\mathcal{A}_{[\eta]}^* = \mathcal{G}_{\eta, u}$. ■

7 An Example: vertical exact magnetic Lagrangian

In this section we present a Lagrangian on the two torus \mathbb{T}^2 such that the β -function is of class C^1 in the open set

$$A = \{(h_1, h_2) \in H_1(\mathbb{T}^2; \mathbb{R}) : h_1 \neq 0\}$$

and is not differentiable at any point outside of A . Let us consider the magnetic Lagrangian on the two torus \mathbb{T}^2 defined by

$$L(x, y, v_1, v_2) = \frac{\|v\|^2}{2} + \langle (0, \cos(2\pi x)), v \rangle$$

where the metric $\|\cdot\|$ is induced by inner product. This type of convex and superlinear Lagrangian is an example of vertical magnetic Lagrangian, apresented in [6], in which the authors were interested in flats of β function. Here we are interested in the differentiability of β and consequently in flats of α .

The Euler-Lagrange flow associated with this Lagrangian is generated by the vector field:

$$X_L : \begin{cases} \dot{x} = v \\ \dot{v} = -2\pi \sin(2\pi x) Jv \end{cases}$$

where J is the 2×2 canonical symplectic matrix. Since the energy function for L is $E(x, y, v) = \frac{1}{2} \|v\|^2$, for each energy level $E > 0$, we can consider the angle φ (with horizontal line) of trajectories of the Euler-Lagrange flow. This means that φ is the new parameter of $v = (v_1, v_2)$:

$$v_1 = \sqrt{2E} \cos \varphi, v_2 = \sqrt{2E} \sin \varphi.$$

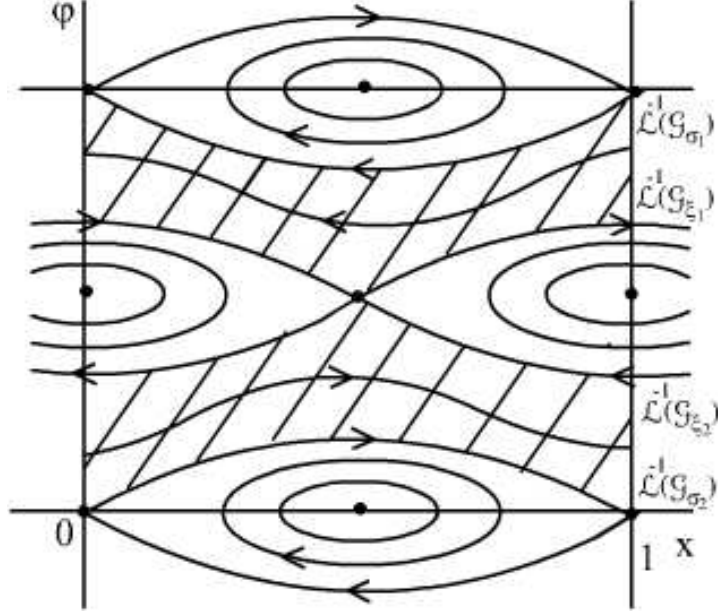


Figure 1: Energy level $E > \frac{1}{2}$

It is easy to see that $H(x, \varphi) = \cos(2\pi x) + \sqrt{2E} \sin \varphi$ is a first integral. The critical points of H are $(0, \frac{\pi}{2})$ -maximum, $(0, -\frac{\pi}{2})$ -saddle, $(\frac{1}{2}, \frac{\pi}{2})$ -saddle and $(\frac{1}{2}, -\frac{\pi}{2})$ -minimum.

Depending of the level of energy, there exist or not invariant Lipschitz Lagrangian graphs. A sufficient condition for existing invariant Lipschitz Lagrangian graphs in the level of energy E is $E > \frac{1}{2}$. In fact, if $|F| < \sqrt{2E} - 1$, the level $H^{-1}(F)$ of H is composed by two graphs. This follows from Implicit Function Theorem. Indeed, $\frac{\partial H}{\partial \varphi} = \sqrt{2E} \cos \varphi = 0$ if and only if $\varphi = \pm \frac{\pi}{2}$. In this case, $\cos(2\pi x) \pm \sqrt{2E} = F$ implies $F \geq \sqrt{2E} - 1$ or $F \leq 1 - \sqrt{2E}$, which contradicts $|F| < \sqrt{2E} - 1$. This also shows that the two graphs of φ_1 and φ_2 with $\varphi_2 = \pi - \varphi_1$, given implicitly by equation

$$\cos(2\pi x) + \sqrt{2E} \sin \varphi = F$$

are invariant for $E > \frac{1}{2}$ and $|F| < \sqrt{2E} - 1$.

The figure 1 describes the projection these graphs in the section $x\varphi$, in the energy level $E > \frac{1}{2}$. Let us consider the closed 1-forms

$$\eta_i = \sqrt{2E} \cos \varphi_i(x) dx + F dy, i = 1, 2, \quad (4)$$

dependent of E and F . Since $\mathcal{L}(x, v) = (x, \langle (v_1, v_2 + \cos(2\pi x)), \cdot \rangle)$, we have $\mathcal{G}_{\eta_i, 0} = \mathcal{L}(x, \sqrt{2E} \cos \varphi_i, \sqrt{2E} \sin \varphi_i)$. These graphs compose a local foliation in $T^*\mathbb{T}^2$, then are absorbing graphs. It follows from Theorem 1 that β is differentiable at any point

of $\partial\alpha([\eta_i])$, where

$$[\eta_i] = \int_0^1 \sqrt{2E} \cos \varphi_i(x) dx + F dy, \quad (5)$$

for all $i = 1, 2$, $E > \frac{1}{2}$ and F with $|F| < \sqrt{2E} - 1$. Moreover, by Proposition 14, we conclude that α function is differentiable at $[\eta_i]$.

Take $F \rightarrow \sqrt{2E} - 1$ by left and $F \rightarrow 1 - \sqrt{2E}$ by right in (5), for each $i = 1, 2$, we obtain four closed 1-forms $\sigma_1, \sigma_2, \xi_1$ and ξ_2 whose graphs are the connections of saddle. Therefore $\mathcal{G}_{\sigma_1,0} \cap \mathcal{G}_{\sigma_2,0} \neq \emptyset$ and the graphs $\mathcal{G}_{\sigma_1,0}$ and $\mathcal{G}_{\sigma_2,0}$ is not absorbing. Moreover, since $\mathcal{G}_{\eta_i,0}$ are absorbing graphs, $\mathcal{A}_{[\eta_i]}^* = \mathcal{G}_{\eta_i,0}$. Then, by semicontinuity of Aubry set, we obtain $\mathcal{A}_{[\sigma_i]} = \mathbb{T}^2$, so $\mathcal{A}_{[\sigma_i]}^* = \mathcal{G}_{\sigma_i,0}$. It follows from ([14], Proposition 6) that $[\sigma_1]$ and $[\sigma_2]$ belong to same flat of α . We obtain, analogously, that $[\xi_1]$ and $[\xi_2]$ belong to same flat of α . This show that β is neither differentiable at points of $\partial\alpha([\sigma_i])$ nor at points of $\partial\alpha([\xi_i])$. Actually, the homology classes which belong to $\partial\alpha([\sigma_i])$ and $\partial\alpha([\xi_i])$ are $\pm\sqrt{2E}(0, 1)$. In fact, the intersection $\mathcal{A}_{[\sigma_1]} \cap \mathcal{A}_{[\sigma_2]}$ is the intersection of two graphs given by implicit equation

$$\cos(2\pi x) + \sqrt{2E} \sin \varphi = \sqrt{2E} - 1.$$

Thence $\mathcal{A}_{[\sigma_1]} \cap \mathcal{A}_{[\sigma_2]}$ is the closed curve $\gamma_1 : t \mapsto \left(\frac{1}{2}, \sqrt{2E}t\right)$. In particular, γ_1 belongs to $\mathcal{M}_{[\sigma_1]} \cap \mathcal{M}_{[\sigma_2]}$ which supports the measure μ_1 with rotation vector equals $\rho(\mu_1) = \sqrt{2E}(0, 1)$. Analogously, $\mathcal{A}_{[\xi_1]} \cap \mathcal{A}_{[\xi_2]}$ is the closed curve $\gamma_2 : t \mapsto \left(0, -\sqrt{2E}t\right)$ which supports a probability measure μ_2 with $\rho(\mu_2) = -\sqrt{2E}(0, 1)$. Therefore we prove that β function is not differentiable at any homology class of the form $(0, h_2)$.

Remark 16 *Since H and E are first integrals, given two constants a and b , the level set $(H, E)^{-1}(a, b) = \{(x, v) : H(x, v) = a, E(x, v) = b\}$ is an absorbing set. Therefore, it follows from Proposition 6 that if $H(\tilde{\mathcal{A}}_c) = F$ for some $c \in H^1(\mathbb{T}^2; \mathbb{R})$, then $(H, E)^{-1}(F, \alpha(c))$ projects onto the whole torus \mathbb{T}^2 .*

By previous remark, for all $c \in H^1(\mathbb{T}^2; \mathbb{R})$ the Aubry set $\tilde{\mathcal{A}}_c$ is contained in an invariant Lipschitz Lagrangian graph. Since the supports of minimizing measures contained in the graphs $\mathcal{G}_{\sigma_i,0}$ and $\mathcal{G}_{\xi_i,0}$ have vector rotation of the form $(0, h_2)$, if $h \in A$ then $\tilde{\mathcal{M}}^h$ is contained in a neighborhood foliated by Lipschitz Lagrangian graphs. It follows from Theorem 15 and Proposition 14 that β is of class C^1 in some neighborhood of h and hence of class C^1 in A .

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